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The structure of Lie derivations on C^*-algebras

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Abstract

We prove that every Lie derivation on a C^*-algebra is in standard form, that is, it can be uniquely decomposed into the sum of a derivation and a centre-valued trace.

1. Introduction

The structure of Lie derivations on C^*-algebras, and on more general Banach algebras, has attracted some attention over the past years. Johnson proved in [8] that every continuous Lie derivation D from a C^*-algebra A into a Banach A-bimodule X can be decomposed as $D = d + \tau$, where $d : A \to X$ is an ordinary derivation and τ is a linear mapping from A into the centre of X. This result was obtained by cohomological methods, namely the concept of symmetric amenability, and in fact holds for symmetrically amenable Banach algebras [8, Theorem 9.2]. In [15] Villena undertook an investigation of the structure of not necessarily bounded Lie derivations on unital Banach algebras. The main result therein states that, if

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$D: A \to A$ is a Lie derivation on a (unital, complex) Banach algebra A, then the canonically induced Lie derivation $D_P: A \to A/P$ decomposes into the sum of a derivation d_P on A/P and a linear functional τ_P on A for almost all primitive ideals P of A. "Almost all" here means that there may exist a finite set of primitive ideals of finite codimension greater than one for which the decomposition fails. However, if D is continuous, then this set must be empty. A drawback of this type of decomposition is that it cannot be lifted to a global decomposition on the level of the algebra A in general. The methods employed to obtain this result combined automatic continuity theory with refinements of algebraic techniques that originate from Herstein's programme of relating Lie derivations and Lie isomorphisms on rings to their associative counterparts, which were pushed very far by Brešar [4]. In [2] Ara and Mathieu developed a theory of local multipliers of C^*-algebras, which parallels the algebraic theory of the symmetric ring of quotients of semiprime rings as initiated by Kharchenko and Martindale. Using methods similar to the algebraic setting but adapted to the situation of C^*-algebras, they obtained the standard decomposition of Lie derivations without boundedness assumptions for all C^*-algebras which are "non-commutative enough" (for the precise meaning, see Section 2); cf. also [10]. In this article we will combine the techniques in [2,15] in order to take a fresh look at the decomposition of a Lie derivation on a C^*-algebra modulo primitive ideals which enables us to put the local constituents together via the Dauns–Hofmann theorem. More precisely we shall prove the following result.

Theorem 1.1. *Let A be a C^*-algebra, and let D be a Lie derivation on A. Then there exist a unique derivation d on A and a unique linear mapping τ from A into the centre of A such that $D = d + \tau$.*

It is easily checked that a Lie derivation which takes its values in the centre vanishes on commutators. As the above mapping τ enjoys this property, we shall refer to it as a *centre-valued trace*. Note also that the uniqueness statement is evident. In fact, let $D = d_1 + \tau_1 = d_2 + \tau_2$ be two decompositions of the Lie derivation D into the sum of a derivation and a centre-valued trace. Then $d_1 - d_2$ is a derivation mapping A into its centre $Z(A)$, wherefore the general Kleinecke–Shirokov theorem, see [5, Theorem 2.7.19], for example, entails that $d_1 - d_2 = 0$. The existence part, however, is somewhat involved but we manage to avoid almost all cohomological devices; merely the basic fact that every abelian C^*-algebra is weakly amenable will be used in Lemma 3.9.

Note that the above theorem implies that the only obstruction to continuity of a Lie derivation on a C^*-algebra lies in the centre-valued trace. Indeed, in Theorem 3.8 below we shall show that, if A has no *bounded* traces, every Lie derivation on A is continuous. However, a comprehensive continuity statement was obtained more generally by Berenguer and Villena [3]: The separating space $\mathfrak{S}(D)$ of every Lie derivation on a semisimple Banach algebra A is contained in the centre of A. This not withstanding, we shall employ some automatic continuity theory related to results by Thomas [14] in Section 2.

2. Prerequisites and terminology

Throughout this paper, we shall use the letters A, B for C^*-algebras, the letters I, J for closed (two-sided) ideals, and P and Q for primitive ideals. The commutator of two elements a, b is $[a, b] = ab - ba$. Together with this (non-associative) product, a C^*-algebra A becomes a Banach–Lie algebra $(A, [\cdot, \cdot])$. A derivation D of this structure is a *Lie derivation* on A, that is, a linear mapping $D : A \to A$ satisfying the identity $D([a, b]) = [D(a), b] + [a, D(b)]$ for all $a, b \in A$. Every (associative) derivation $d : A \to A$ is clearly a Lie derivation on A, as is every linear mapping $\tau : A \to Z(A)$, the centre of A, with the property $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We call such a mapping τ a *centre-valued trace* on A. We say that a Lie derivation is in *standard form* if $D = d + \tau$ for some derivation d on A and some centre-valued trace τ. As observed in Section 1, if such a decomposition exists, it is necessarily unique.

The main question on the structure of Lie derivations has been under what conditions every Lie derivation is in standard form. Every centre-valued trace is evidently a "trivial" Lie derivation, and thus the existence of a standard form describes the difference between Lie derivations and associative derivations in a precise way.

Suppose that I is a closed ideal in A, and denote by $\pi_I : A \to A/I$ the canonical quotient mapping. Every Lie derivation $D : A \to A$ yields a Lie derivation $D_I : A \to A/I$ via $D_I = \pi_I \circ D$. However, D_I in general cannot be dropped to a Lie derivation from A/I to A/I, since I may not be invariant under D. On the other hand, if d is a derivation on A, it induces a derivation $d_I : A/I \to A/I$, as $d(I) \subseteq I$. (Consequently, we may without danger of ambiguity denote both the derivations $\pi_I \circ d : A \to A/I$ and $d_I \circ \pi_I : A/I \to A/I$ with the same symbol d_I.) Several of the problems in the structure theory of Lie derivations arise from this fundamentally different behaviour of Lie derivations and their associative counterparts.

A fruitful approach to the study of the standard form of Lie derivations in the algebraic, that is, ring theoretic setting has been the use of polynomial identities and the extended centroid of a semiprime ring. In this spirit, Brešar proved in [4, Theorem 5] that every Lie derivation on a prime ring of characteristic different from 2 which does not satisfy the standard polynomial identity S_4 is in standard form; however, the associative derivation may take on its values in the central closure and the trace may map into the extended centroid, which in general are much larger than the original ring and its centre, respectively. We refer to [2,4,15] for further references pertinent to this topic.

The theory of local multipliers of a C^*-algebra A developed in [2] provides us with a C^*-algebraic analogue of the algebraic theory alluded to above. Applied to Lie derivations, it bypasses the transition to quotient algebras, albeit the need for an assumption of a certain degree of non-commutativity persists. Let K_1 denote the usual closed commutator ideal, that is, the closed ideal generated by all commutators in a C^*-algebra A. Let K_2 denote the closed ideal generated by all expressions of the form

$$[x^2, y]z[x, y] - [x, y]z[x^2, y] \quad (x, y, z \in A).$$

Evidently, $K_2 \subseteq K_1$ and hence the reverse inclusion $K_1^\perp \subseteq K_2^\perp$ for their annihilators always holds. (Where $I^\perp = \{x \in A \mid xI = Ix = 0\}$ for an ideal I of A.) If A is the algebra of 2×2 matrices over an abelian C^*-algebraic, then $K_1^\perp = 0$ whereas $K_2^\perp = A$. In this situation, A satisfies S_4 [2, Theorem 6.1.7], which is the case we have to avoid.

The following result was proved in [2, Theorem 6.4.1] for the case $I = 0$ but the proof can easily be adapted to the more general situation needed in the sequel. We shall use the following terminology, see [2, Section 3.2]. Every unital C^*-algebra B is contained in a (possibly) larger unital C^*-algebra ${}^{c}B$, called the *bounded central closure* of B. This C^*-algebra has the following property. Let J be a closed essential ideal in ${}^{c}B$. Then the centre $Z(M(J))$ of the multiplier algebra $M(J)$ of J coincides with $Z({}^{c}B)$. We say that B is *boundedly centrally closed* if $B = {}^{c}B$.

Theorem 2.1. *Let A be a unital C^*-algebra, and let D be a Lie derivation on A. Suppose that I is a closed ideal of A and set $B = A/I$. If $K_1^\perp = K_2^\perp$ in B, then there exist a derivation d_I from A into ${}^{c}B$ and a linear mapping τ_I from A into the centre of ${}^{c}B$ such that $D_I = d_I + \tau_I$.*

The reason why the original argument can be extended to cover this situation is the following. Although we are now dealing with module derivations, the module operation is a very special one. In fact, $a \cdot (x + I) = \pi_I(a)\pi_I(x)$ and $(x + I) \cdot a = \pi_I(x)\pi_I(a)$ for all $a, x \in A$. Therefore all identities we have to control are actually identities in the quotient C^*-algebra A/I and thus can be dealt with as before.

Corollary 2.2. *Let A be a unital C^*-algebra, and let D be a Lie derivation on A. For each primitive ideal P of A of codimension greater than 4 there exist a unique derivation d_P from A into A/P and a unique linear functional τ_P on A such that*

$$D_P = d_P + \pi_P(1)\tau_P.$$

Proof. Since the quotient C^*-algebra A/P is prime, every non-zero ideal is essential. Since every non-zero closed ideal J in A/P is a prime C^*-algebra itself, $Z(M(J)) = \mathbb{C} = Z(A/P)$ and A/P is boundedly centrally closed. By assumption, A/P acts faithfully and irreducibly on a Hilbert space of dimension at least 3. Therefore, $K_2(A/P) \neq 0$ [2, Proposition 6.1.4] which implies that $K_2^\perp = 0$. Theorem 2.1 thus yields the existence of the derivation d_P and the linear functional τ_P, and the uniqueness statement is clear. □

Notation. In the future, we will identify $\mathbb{C}$ with $\mathbb{C}1$ whenever convenient and hence drop the $\pi_P(1)$ in front of the trace.

Remark 2.3. Let D be a Lie derivation on a unital C^*-algebra A. For every primitive ideal P of A of codimension 1, each derivation d_P from A into A/P vanishes. Thus there is a unique linear functional τ_P on A such that $D_P = \tau_P$. Note that no primitive ideals of codimension 2 or 3 exist.

By the above considerations, we are left with primitive ideals P of codimension 4, that is, with primitive quotients A/P equal to $M_2(\mathbb{C})$. For this case, the theory of polynomial identities cannot contribute anything, mainly since $K_2 = 0$ in this situation. In order to obtain the standard decomposition $D_P = d_P + \tau_P$ nevertheless, we have to resort to automatic continuity theory. The two basic principles we use are the gliding hump argument and the invariance argument, which can be found in [14, Proposition 1.3] and in [14, Lemma 1.1] and [15, Lemma 3.1], respectively.

Lemma 2.4 (Gliding hump argument). *Let X and Y be Banach spaces, let $(R_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear operators from X to itself, and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear operators from Y to Banach spaces Y_n. If T is a linear operator from X to Y such that $S_nTR_1\cdots R_m$ is continuous for $m>n$, then $S_nTR_1\cdots R_n$ is continuous for sufficiently large n.*

Suppose that $T: X\to Y$ is a linear mapping between the Banach spaces X and Y. The possible discontinuity of T is measured by the *separating space* $\mathfrak{S}(T)$ defined by

$$\mathfrak{S}(T) = \{y\in Y \mid \exists\, (x_n)_{n\in\mathbb{N}} \subseteq X \text{ such that } x_n\to 0 \text{ and } T(x_n)\to y\}.$$

By the closed graph theorem, $\mathfrak{S}(T) = 0$ if and only if T is bounded. For a closed subspace M of X, we denote by π_M the canonical mapping $X\to X/M$.

Lemma 2.5 (Invariance argument). *Let M be a closed subspace of a Banach space X, and let T be a linear operator from X to itself such that $\mathfrak{S}(T^n)\subseteq M$ for all $n\in\mathbb{N}$. Then there exists a positive constant γ such that $\|\pi_M T^n\|\leqslant\gamma^n$ for all $n\in\mathbb{N}$.*

Let $\mathrm{Prim}(A)$ denote the set of all primitive ideals of the C^*-algebra A, and let $\mathfrak{P}$ be the set of all those $P\in\mathrm{Prim}(A)$ for which the decomposition $D_P = d_P + \tau_P$ holds. Our first aim in the next section, in which we shall give the proof of Theorem 1.1, will be to show that $\mathfrak{P} = \mathrm{Prim}(A)$. Here we show that, for every $P\in\mathfrak{P}$, we necessarily have $[\mathfrak{S}(D^n), A]\subseteq P$ for each $n\in\mathbb{N}$. Since the derivation d_P is continuous, it follows that $\pi_P(\mathfrak{S}(D))\subseteq Z(A/P)$ and thus $[\mathfrak{S}(D), A]\subseteq P$. Suppose that $[\mathfrak{S}(D^n), A]\subseteq P$ for some $n\geqslant 1$. For every $a\in A$ we have

$$\mathrm{ad}(a)D^{n+1} = D\,\mathrm{ad}(a)D^n - \mathrm{ad}(D(a))D^n,$$

where $\mathrm{ad}(a)$ denotes the inner derivation $\mathrm{ad}(a)(x) = [a,x]$ for all $x\in A$. Therefore

$$\pi_P\,\mathrm{ad}(a)D^{n+1} = d_P\,\mathrm{ad}(a)D^n + \tau_P\,\mathrm{ad}(a)D^n - \pi_P\,\mathrm{ad}(D(a))D^n.$$

Since $[\mathfrak{S}(D^n), A]\subseteq P$, it follows that both $d_P\,\mathrm{ad}(a)D^n$ and $\pi_P\,\mathrm{ad}(D(a))D^n$ are continuous. Hence, $\mathfrak{S}(\pi_P\,\mathrm{ad}(a)D^{n+1})\subseteq Z(A/P)$, which entails that $[\pi_P(\mathfrak{S}(D^{n+1})), \pi_P(a)]\subseteq Z(A/P)$. The Kleinecke–Shirokov theorem now shows that $[\pi_P(\mathfrak{S}$

$(D^{n+1})), \pi_P(a)]$ consists of quasinilpotent central elements of A/P, which clearly forces $[\mathfrak{S}(D^{n+1}), a] \subseteq P$, as required.

Consequently, we will next study the condition $[\mathfrak{S}(D^n), A] \subseteq P$. Lemmas 2.6 and 2.8 below were obtained in a more general formulation in [15, Lemma 2.6] and [15, Lemma 3.3], respectively, but we have included slightly simplified proofs for the case of C^*-algebras for completeness and the convenience of the reader.

Lemma 2.6. *The set of those primitive ideals P of A of codimension 4 for which the condition*

$$[\mathfrak{S}(D^n), A] \subseteq P \quad (n \in \mathbb{N})$$

fails is finite.

Proof. Suppose towards a contradiction that we can find a sequence $(P_n)_{n\in\mathbb{N}}$ of pairwise different primitive ideals of A which have codimension 4 and, for every $n \in \mathbb{N}$, there is a natural number k_n such that $[\mathfrak{S}(D^{k_n}), A] \nsubseteq P_n$ and $[\mathfrak{S}(D^k), A] \subseteq P_n$ if $k < k_n$.

Note that $\mathfrak{S}(D^{k_n})$ is a Lie ideal of A for each $n \in \mathbb{N}$. Indeed, let $a \in \mathfrak{S}(D^{k_n})$ and $b \in A$. Then there exists a sequence $(a_j)_{j\in\mathbb{N}}$ in A such that $\lim_{j\to\infty} a_j = 0$ and $\lim_{j\to\infty} D^{k_n}(a_j) = a$. Since

$$D^{k_n}([a_j, b]) = [D^{k_n}(a_j), b] + \sum_{i=1}^{k_n} \binom{k_n}{i} [D^{k_n - i}(a_j), D^i(b)]$$

and D^i is continuous for $i < k_n$, we see that $\lim_{j\to\infty} D^{k_n}([a_j, b]) = \lim_{j\to\infty} [D^{k_n}(a_j), b] = [a, b]$.

We proceed to show that there exist sequences (a_n), (b_n), and (c_n) in A satisfying the following conditions:

(i) $a_{n+1} \in P_1 \cap \cdots \cap P_n$ for each $n \in \mathbb{N}$;
(ii) $\operatorname{ad}(b_n)\operatorname{ad}(a_1)\cdots\operatorname{ad}(a_n)(c_n) \notin P_n$ for each $n \in \mathbb{N}$.

By hypothesis, for each $n \in \mathbb{N}$, A/P_n is isomorphic to the matrix algebra $M_2(\mathbb{C})$. Let $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$. It is readily checked that $\operatorname{ad}(u)(v) = 2iv$. For every $n \in \mathbb{N}$, take $b_n, c_n \in A$ such that $b_n + P_n = u$ and $c_n + P_n = v$. Since the map $a \mapsto (a + P_1, \ldots, a + P_n)$ from A to $A/P_1 \oplus \cdots \oplus A/P_n$ is surjective, we deduce that, for every $k \in \mathbb{N}$, there exists $w_k \in A$ such that $w_k + P_k = u$ and $w_k + P_j = 0 + P_j$ for $j < k$. Let $\lambda_1 \in \mathbb{C}$ be such that $0 < \|\lambda_1 w_1\| < 2^{-1}$. We have

$$\operatorname{ad}(u)\operatorname{ad}(\lambda_1 w_1 + P_1)(v) = \lambda_1 \operatorname{ad}(u)\operatorname{ad}(u)(v) = \lambda_1 (2i)^2 v \neq 0.$$

Suppose that $\lambda_1, \ldots, \lambda_n$ have been chosen such that $\|\lambda_k w_k\| < 2^{-k}$ and

$$\operatorname{ad}(u)\operatorname{ad}(\lambda_1 w_1 + \cdots + \lambda_k w_k + P_k)$$
$$\cdots \operatorname{ad}(\lambda_i w_i + \cdots + \lambda_k w_k + P_k) \cdots \operatorname{ad}(\lambda_k w_k + P_k)(v) \neq 0$$

for $k = 1, \dots, n$. We consider the polynomial $p(\lambda)$ with coefficients in A given by

$$\begin{aligned} p(\lambda) &= \mathrm{ad}(u)\,\mathrm{ad}(\lambda_1 w_1 + \cdots + \lambda_n w_n + \lambda w_{n+1} + P_{n+1}) \\ &\quad \cdots \mathrm{ad}(\lambda_i w_i + \cdots + \lambda_n w_n + \lambda w_{n+1} + P_{n+1}) \\ &\quad \cdots \mathrm{ad}(\lambda_n w_n + \lambda w_{n+1} + P_{n+1})\,\mathrm{ad}(\lambda w_{n+1} + P_{n+1})(v) \\ &= r_0 + \lambda r_1 + \cdots + \lambda^n r_n + \lambda^{n+1} r_{n+1}. \end{aligned}$$

It is easy to check that $r_{n+1} = (2i)^{n+2} v$ and therefore $p(\lambda)$ is a non-zero polynomial which has at most $n+1$ pairwise different roots. Consequently, there exists a complex number λ_{n+1} such that $||\lambda_{n+1} w_{n+1}|| < 2^{-(n+1)}$ and

$$\mathrm{ad}(u)\,\mathrm{ad}(\lambda_1 w_1 + \cdots + \lambda_{n+1} w_{n+1} + P_{n+1}) \cdots \mathrm{ad}(\lambda_{n+1} w_{n+1} + P_{n+1})(v) \neq 0.$$

For every $n \in \mathbb{N}$ we define $a_n = \sum_{k=n}^{\infty} \lambda_k w_k$. The sequences (a_n), (b_n), and (c_n) satisfy our requirements.

Let $I_n = \{a \in A \mid [a, A] \subseteq P_n\}$, $n \in \mathbb{N}$, and let $\pi_n = \pi_{I_n}$ be the quotient map from A onto A/I_n. Since $A/I_n \cong (A/P_n)/Z(A/P_n) \cong M_2(\mathbb{C})/Z(M_2(\mathbb{C}))$ we conclude that A/I_n is a simple Lie algebra. Let $(S_n)_{n \in \mathbb{N}}$ be the sequence of continuous linear operators from A to itself given by $S_n = \mathrm{ad}(a_n)$ for all $n \in \mathbb{N}$, and let $(R_n)_{n \in \mathbb{N}}$ be the sequence of continuous linear operators from A to A/I_n given by $R_n = \pi_n D^{k_n - 1}$. For all $m, n \in \mathbb{N}$, we have

$$\begin{aligned} R_n D S_1 \cdots S_m &= \pi_n D^{k_n}\,\mathrm{ad}(a_1) \cdots \mathrm{ad}(a_m) \\ &= \pi_n \sum_{i_1 + \cdots + i_{m+1} = k_n} \frac{k_n!}{i_1! \cdots i_{m+1}!} \mathrm{ad}(D^{i_1}(a_1)) \cdots \mathrm{ad}(D^{i_m}(a_m)) D^{i_{m+1}} \\ &= \sum_{i_1 + \cdots + i_{m+1} = k_n} \frac{k_n!}{i_1! \cdots i_{m+1}!} \mathrm{ad}(\pi_n D^{i_1}(a_1)) \cdots \mathrm{ad}(\pi_n D^{i_m}(a_m)) \pi_n D^{i_{m+1}} \\ &= \sum_{\substack{i_1 + \cdots + i_{m+1} = k_n \\ i_{m+1} < k_n}} \frac{k_n!}{i_1! \cdots i_{m+1}!} \mathrm{ad}(\pi_n D^{i_1}(a_1)) \cdots \mathrm{ad}(\pi_n D^{i_m}(a_m)) \pi_n D^{i_{m+1}} \\ &\quad + \mathrm{ad}(\pi_n(a_1)) \cdots \mathrm{ad}(\pi_n(a_m)) \pi_n D^{k_n}, \end{aligned}$$

which coincides with the continuous linear operator

$$\sum_{\substack{i_1 + \cdots + i_{m+1} = k_n \\ i_{m+1} < k_n}} \frac{k_n!}{i_1! \cdots i_{m+1}!} \mathrm{ad}(\pi_n D^{i_1}(a_1)) \cdots \mathrm{ad}(\pi_n D^{i_m}(a_m)) \pi_n D^{i_{m+1}}$$

for $m > n$, since $a_m \in P_n$. The gliding hump argument (Lemma 2.4) shows that the operator

$$\pi_n D^{k_n}\,\mathrm{ad}(a_1)\cdots\mathrm{ad}(a_n)$$

is continuous for sufficiently large n; thus the operator $\mathrm{ad}(\pi_n(a_1))\cdots\mathrm{ad}(\pi_n(a_n))\pi_n D^{k_n}$ is bounded as well. Consequently,

$$\mathrm{ad}(\pi_n(a_1))\cdots\mathrm{ad}(\pi_n(a_n))\pi_n(\mathfrak{S}(D^{k_n})) = 0.$$

Since A/I_n is a simple Lie algebra and $\pi_n(\mathfrak{S}(D^{k_n}))$ is a non-zero Lie ideal of A/I_n, we have $\pi_n(\mathfrak{S}(D^{k_n})) = A/I_n$. We thus find that $\mathrm{ad}(\pi_n(a_1))\cdots\mathrm{ad}(\pi_n(a_n))(A/I_n) = 0$, which entails that $\mathrm{ad}(b_n)\mathrm{ad}(a_1)\cdots\mathrm{ad}(a_n)(c_n) \in P_n$, a contradiction. This proves the lemma. □

The next lemma, due to Zassenhaus, is a standard tool in Lie theory, see [7, Theorem 6 in Chapter III, Section 6].

Lemma 2.7. *Let L be a finite-dimensional simple complex Lie algebra. Then every derivation of L is inner.*

Lemma 2.8. *Suppose that P is a primitive ideal in A of codimension 4 such that $[\mathfrak{S}(D^n), A] \subseteq P$ for all $n \in \mathbb{N}$. Then there exist a derivation d_P from A into A/P and a linear functional τ_P on A such that $D_P = d_P + \tau_P$.*

Proof. Set $I = \{a \in A \mid [a, A] \subseteq P\}$; as in the preceding lemma, A/I is a simple Lie algebra. Assume that $D(I) \nsubseteq I$. Since $\pi_I D(I)$ is a non-zero Lie ideal of A/I, it follows that $\pi_I D(I) = A/I$. On the other hand, for all $a \in A$ and $n \in \mathbb{N}$, we have

$$D^n\,\mathrm{ad}(a)^n = \sum_{i_1+\cdots+i_{n+1}=n} \frac{n!}{i_1!\cdots i_{n+1}!}\,\mathrm{ad}(D^{i_1}(a))\cdots\mathrm{ad}(D^{i_n}(a))D^{i_{n+1}}.$$

If $a \in I$ and $n \in \mathbb{N}$, then $\pi_I D^n\,\mathrm{ad}(a)^n = n!\,\pi_I\,\mathrm{ad}(D(a))^n$. Since $\mathfrak{S}(D^n) \subseteq I$ for all $n \in \mathbb{N}$, Lemma 2.5 gives a constant $\gamma > 0$ such that $\|\pi_I D^n\| \leqslant \gamma^n$ for each $n \in \mathbb{N}$. It follows that

$$n!\|\pi_I\,\mathrm{ad}(D(a))^n\| \leqslant \|\pi_I D^n\,\mathrm{ad}(a)^n\| \leqslant \|\pi_I D^n\|\,\|\mathrm{ad}(a)^n\| \leqslant \gamma^n \|\mathrm{ad}(a)\|^n.$$

Thus $\lim_{n\to\infty} \|\pi_I \mathrm{ad}(D(a))^n\|^{1/n} = 0$ for all $a \in I$. Since

$$\|\mathrm{ad}(\pi_I D(a))^n\|^{1/n} = \|\pi_I\,\mathrm{ad}(D(a))^n\|^{1/n},$$

we conclude that $\lim\|\,\mathrm{ad}(\pi_I D(a))^n\|^{1/n} = 0$ for each $a \in I$. As $\pi_I D(I) = A/I$, it follows that $\mathrm{ad}(x)$ is a quasinilpotent operator on A/I for all $x \in A/I$. Once again, we consider the matrices $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$ and take $r, s \in A$ such that

$r + P = u$ and $s + P = v$. Since $\mathrm{ad}(u)(v) = 2iv$, we have $\mathrm{ad}(r + I)(s + I) = 2is + I$ and therefore $\mathrm{ad}(r + I)$ is not a quasinilpotent operator, a contradiction.

From $D(I) \subseteq I$ we conclude that D drops to a derivation D_I of the finite-dimensional simple Lie algebra A/I defined by $D_I(a + I) = D(a) + I$ for each $a \in A$. By Lemma 2.7, D_I is an inner derivation, so there is $a \in A$ such that $D(b) - \mathrm{ad}(a)(b) \in I$ for all $b \in A$. As this implies that

$$\pi_P D(b) - \mathrm{ad}(\pi_P(a))(\pi_P(b)) \in \pi_P(I) = Z(A/P) = \mathbb{C}\pi_P(1),$$

we can take $d_P = \pi_P\, \mathrm{ad}(a)$ and define $\tau_P(b) = D_P(b) - d_P(b)$ for each $b \in A$ in order to obtain the decomposition $D_P = d_P + \tau_P$. $\square$

3. Lie derivations on C^*-algebras

Our objective in this section is to prove Theorem 1.1, our main theorem. Although a Lie derivation need not leave primitive ideals invariant, we first obtain the standard form in each irreducible representation in Theorem 3.1. The proof of this result uses all the prerequisites provided in Section 2. We then will "glue the local decompositions together". This step is based on the Dauns–Hofmann theorem as we shall concentrate on the local parts of the trace (which lie in the complex numbers) but is still rather involved and uses Theorem 2.1 one more time. Therefore, we shall state intermediate results which are of interest in their own as theorems, despite the fact that they all are subsumed under the main theorem.

Theorem 3.1. *Let A be a unital C^*-algebra, and let D be a Lie derivation on A. For every primitive ideal P of A there exist a unique derivation d_P from A into A/P and a unique linear functional τ_P on A such that $D_P = d_P + \tau_P$.*

Proof. Recall that $\mathfrak{P}$ denotes the set of all primitive ideals P of A for which the statement in the theorem holds. Let

$$I = \bigcap_{P \in \mathfrak{P}} P$$

(we write $I = A$ in the case where $\mathfrak{P} = \emptyset$). In view of Corollary 2.2, Lemmas 2.6 and 2.8, $\mathrm{Prim}(A) \backslash \mathfrak{P}$ is finite and consists of primitive ideals of codimension 4. Write $\mathrm{Prim}(A) \backslash \mathfrak{P} = \{P_1, \dots, P_m\}$ and note that $I \nsubseteq P_k$ for all $1 \leqslant k \leqslant m$. In fact, if $P \in \mathrm{Prim}(A)$ has codimension 4 and $I \subseteq P$ then $[\mathfrak{S}(D^n), A] \subseteq P$ for all n, by the remarks preceding Lemma 2.6. Thus, by Lemma 2.8, $P \in \mathfrak{P}$.

Since the map $a \mapsto (a + P_1, \dots, a + P_m)$ from I into $A/P_1 \oplus \cdots \oplus A/P_m$ is injective, $\dim I < \infty$. If $a \in I$, then $\dim(\mathrm{ad}(a)(A)) < \infty$ and therefore $D\, \mathrm{ad}(a)$ is continuous. On the other hand,

$$D\, \mathrm{ad}(a) = \mathrm{ad}(D(a)) + \mathrm{ad}(a) D,$$

which shows that $\mathrm{ad}(a)D$ is continuous and hence $\mathrm{ad}(a)(\mathfrak{S}(D)) = 0$. Consequently, $[\mathfrak{S}(D), I] = 0$. Since $I \not\subseteq P_k$, we have

$$0 = \pi_{P_k}([\mathfrak{S}(D), I]) = [\pi_{P_k}(\mathfrak{S}(D)), \pi_{P_k}(I)]$$

$$= [\pi_{P_k}(\mathfrak{S}(D)), A/P_k] = \pi_{P_k}([\mathfrak{S}(D), A]),$$

which shows that

$$[\mathfrak{S}(D), A] \subseteq P_k$$

for each $k = 1, \dots, m$. This together with the remarks preceding Lemma 2.6 entails that

$$[\mathfrak{S}(D), A] \subseteq I \cap P_1 \cap \dots \cap P_m = 0.$$

Supposing that $[\mathfrak{S}(D^k), A] = 0$ for all $1 \leqslant k \leqslant n$, we prove that $[\mathfrak{S}(D^{n+1}), A] = 0$. For each $a \in I$, we have

$$D^{n+1}\,\mathrm{ad}(a) = \sum_{k=0}^{n+1} \binom{n+1}{k} \mathrm{ad}(D^{n+1-k}(a)) D^k.$$

Since $[\mathfrak{S}(D^k), A] = 0$ for $k = 1, \dots, n$, it follows that $\mathrm{ad}(D^{n+1-k}(a))D^k$ is continuous for $k = 1, \dots, n$. On the other hand, $D^{n+1}\,\mathrm{ad}(a)$ is continuous, since $\dim(\mathrm{ad}(a)(A)) < \infty$. Consequently, $\mathrm{ad}(a)D^{n+1}$ is continuous. From this we deduce that

$$\mathrm{ad}(a)(\mathfrak{S}(D^{n+1})) = 0,$$

and thus $[\mathfrak{S}(D^{n+1}), I] = 0$. Since $I \not\subseteq P_k$, we have

$$0 = \pi_{P_k}([\mathfrak{S}(D^{n+1}), I]) = [\pi_{P_k}(\mathfrak{S}(D^{n+1})), \pi_{P_k}(I)]$$

$$= [\pi_{P_k}(\mathfrak{S}(D^{n+1})), A/P_k] = \pi_{P_k}([\mathfrak{S}(D^{n+1}), A]),$$

which shows that

$$[\mathfrak{S}(D^{n+1}), A] \subseteq P_k \quad (1 \leqslant k \leqslant m).$$

As before we find that

$$[\mathfrak{S}(D^{n+1}), A] \subseteq I \cap P_1 \cap \dots \cap P_m = 0.$$

Consequently, $[\mathfrak{S}(D^n), A] = 0$ for all $n \in \mathbb{N}$, wherefore $\mathrm{Prim}(A) \backslash \mathfrak{P} = \emptyset$ by Lemma 2.8. Finally, suppose that $D_P = d_P + \tau_P = d'_P + \tau'_P$ for some $P \in \mathrm{Prim}(A)$, where d_P and d'_P are derivations from A into A/P and τ_P and τ'_P map into the centre of A/P. Then $d_P - d'_P$ is a derivation from A into $Z(A/P) = \mathbb{C}$. Therefore $d_P = d'_P$, hence $\tau_P = \tau'_P$ as well. $\square$

Remark 3.2. Invoking the rather involved analysis for general Banach algebras carried out in [15], essentially the same arguments as in the proof of Theorem 3.1 show that the statement extends to semisimple unital Banach algebras.

We are now going to put the standard decompositions of the Lie derivation D obtained in irreducible images together to a decomposition on the level of the C^*-algebra A. To this end, we need, for each $a \in A$, an element $\tau(a) \in A$ such that

$$\pi_P(\tau(a)) = \tau_P(a) \quad (P \in \mathrm{Prim}(A)).$$

Such an element is necessarily unique and belongs to the centre of A. The Dauns–Hofmann theorem allows us to state the following criterion for the existence of $\tau(a)$.

In the sequel τ_P, $P \in \mathrm{Prim}(A)$ will always denote the unique functional on the unital C^*-algebra A obtained in Theorem 3.1.

Proposition 3.3. *The following conditions on an element a in a unital C^*-algebra A are equivalent*:

(a) *There exists $\tau(a) \in Z(A)$ such that $\pi_P(\tau(a)) = \tau_P(a)$ for all $P \in \mathrm{Prim}(A)$.*
(b) *The function $P \mapsto \tau_P(a)$ from $\mathrm{Prim}(A)$ into $\mathbb{C}$ is continuous.*

Proof. This is immediate from the Dauns–Hofmann theorem as stated, for example, in [2, Theorem 1.2.28]. $\square$

Therefore we will henceforth focus on condition (b) in the above proposition.

Lemma 3.4. *Let $P, Q \in \mathrm{Prim}(A)$ and $a, b \in A$. Then the following assertions hold*:

(i) $\tau_P = \tau_Q$ *whenever* $P \subseteq Q$.
(ii) $D_P(ab) - D_P(a)\pi_P(b) - \pi_P(a)D_P(b) = -\tau_P(a)\pi_P(b) - \tau_P(b)\pi_P(a) + \tau_P(ab)$.
(iii) $\tau_P([a,b]) = 0$.

Proof. Suppose that $P \subseteq Q$ and let Φ be the epimorphism from A/P onto A/Q defined by $\Phi\pi_P = \pi_Q$, with kernel Q/P. Since Q/P is d_P-invariant, d_P drops to a derivation $d_\Phi : A/Q \to A/Q$ defined by $d_\Phi \pi_Q = \Phi d_P \pi_P$. Note that, for each $x \in A$, $\Phi(\tau_P(x)) = \tau_P(x)\Phi(\pi_P(1)) = \tau_P(x)\pi_Q(1)$. By Theorem 3.1, we thus obtain that

$$d_Q\pi_Q + \tau_Q = \pi_Q D = \Phi\pi_P D = \Phi(d_P\pi_P + \tau_P) = d_\Phi\pi_Q + \tau_P$$

so that the uniqueness statement yields $d_\Phi\pi_Q = d_Q\pi_Q$ and $\tau_P = \tau_Q$. This proves assertion (i). Towards (ii), we have

$$\begin{aligned} D_P(ab) &= d_P(\pi_P(ab)) + \tau_P(ab) \\ &= d_P(\pi_P(a))\pi_P(b) + \pi_P(a)d_P(\pi_P(b)) + \tau_P(ab) \end{aligned}$$

$$= (D_P(a) - \tau_P(a))\pi_P(b) + \pi_P(a)(D_P(a) - \tau_P(b)) + \tau_P(ab)$$

$$= \pi_P(D(a)b + aD(b)) - \tau_P(a)\pi_P(b) - \tau_P(b)\pi_P(a) + \tau_P(ab).$$

From (ii) we now find that

$$\begin{aligned}\tau_P(ab - ba) &= \pi_P(D(ab) - D(a)b - aD(b) + \tau_P(a)b + \tau_P(b)a)\\ &\quad - \pi_P(D(ba) - D(b)a - bD(a) + \tau_P(b)a + \tau_P(a)b)\\ &= \pi_P(D([a,b]) - [D(a),b] - [a,D(b)]) = 0,\end{aligned}$$

which yields assertion (iii). □

Let A_D be the set of those elements $a \in A$ for which there exists $\tau(a) \in A$ such that $\pi_P(\tau(a)) = \tau_P(a)$ for all $P \in \mathrm{Prim}(A)$. This set has some useful properties described in the next result.

Proposition 3.5. *For every unital C^*-algebra A, the set A_D is a subalgebra of A containing both $Z(A)$ and $[A, A]$. The mapping $d : A_D \to A$ defined by $d = D - \tau$ is a derivation.*

Proof. Let $a, b \in A_D$ and $\alpha, \beta \in \mathbb{C}$. For every $P \in \mathrm{Prim}(A)$, we have

$$\pi_P(\alpha\tau(a) + \beta\tau(b)) = \alpha\tau_P(a) + \beta\tau_P(b) = \tau_P(\alpha a + \beta b),$$

which shows that $\alpha a + \beta b \in A_D$ and that $\tau(\alpha a + \beta b) = \alpha\tau(a) + \beta\tau(b)$. Lemma 3.4(ii) entails that

$$\pi_P(D(ab) - D(a)b - aD(b) + \tau(a)b + \tau(b)a) = \tau_P(ab) \quad (P \in \mathrm{Prim}(A)),$$

wherefore $ab \in A_D$. If $a \in Z(A)$, then $\pi_P(a) \in \mathbb{C}\pi_P(1)$ and so $d_P(\pi_P(a)) = 0$ for each $P \in \mathrm{Prim}(A)$. From this we deduce that $\tau_P(a) = \pi_P(D(a))$ for all P, whence $a \in A_D$. Lemma 3.4(iii) yields $\tau_P([A, A]) = 0$ for each $P \in \mathrm{Prim}(A)$ and thus, $[A, A] \subseteq A_D$.

Re-writing identity (ii) in Lemma 3.4 yields

$$(D - \tau)(ab) - (D - \tau)(a)b - a(D - \tau)(b) \in P \quad (P \in \mathrm{Prim}(A)),$$

entailing that d is a derivation on A_D. □

This result shows that A_D is the natural domain of both the centre-valued trace τ and the derivation d, which we want to be defined everywhere. It is thus not surprising that we can obtain several special results already at this stage.

Theorem 3.6. *Let A be a von Neumann algebra, and let D be a Lie derivation on A. Then there exist a unique derivation d on A and a unique linear mapping τ from A into the centre of A such that $D = d + \tau$.*

Proof. It is well known that A can be decomposed into the direct sum of a finite and a properly infinite von Neumann algebra [9, Proposition 6.3.7]. By Fack and de la Harpe [6], every element in the kernel of the canonical central trace in a finite von Neumann algebra is the sum of ten commutators. Thus every element in the finite part is the sum of a central element and a sum of commutators. On the other hand, every element in a properly infinite von Neumann algebra is the sum of two commutators [12]. Consequently, A is the linear span of $Z(A)$ and $[A, A]$. The statement now follows from Proposition 3.5. $\square$

Remark 3.7. Theorem 3.6 belongs to Miers who obtained it in [11] by very different methods. We believe that our approach is by far more conceptual. In particular, it shows that every Lie derivation on a properly infinite von Neumann algebra is indeed a derivation. Using Johnson's original ideas from [8] and the theory of functional identities, Theorem 3.6 is extended to Lie derivations from von Neumann algebras into Banach bimodules in [1].

A result analogous to the one for properly infinite von Neumann algebras can be obtained for C^*-algebras.

Theorem 3.8. *Let A be a unital C^*-algebra without tracial states, and let D be a Lie derivation on A. Then D is a derivation.*

Proof. By Pop [13], there exists an integer $p \geqslant 2$ such that every element of A can be expressed as a sum of p commutators. Thus Proposition 3.5 entails that $A_D = A$. Since τ vanishes on commutators, it follows that D is indeed a derivation. $\square$

From Theorem 3.6 we can also derive the main theorem for *continuous* Lie derivations. To this end, we need the following auxiliary result, which will be used later again.

Lemma 3.9. *Let A be a unital C^*-algebra, X be a Banach A-bimodule, d be a derivation from A into X, and let τ be a linear mapping from A into the centre of X. Suppose that $d + \tau$ maps into a closed A-subbimodule M of X. Then d maps into M and τ maps into the centre of M.*

Proof. Let Φ be the quotient map from X onto the quotient Banach A-bimodule $Y = X/M$. Since $(d + \tau)(A) \subseteq M$ we have $0 = \Phi \circ (d + \tau) = \Phi \circ d + \Phi \circ \tau$ and so $\Phi \circ d = -\Phi \circ \tau$. Since τ maps into the centre $Z(X)$ of X, we see that $\Phi \circ \tau$ maps into the centre of Y. As every abelian C^*-algebra is weakly amenable [5, Corollary 5.3.6], the restriction of the derivation $\Phi \circ d : A \to Z(Y)$ vanishes on every abelian C^*-subalgebra, thus $\Phi \circ d = 0$. As a result, $d(A) \subseteq M$ and $\tau(A) \subseteq Z(M)$. $\square$

Theorem 3.10. *Let A be a C^*-algebra, and let D be a continuous Lie derivation on A. Then there exist a unique derivation d and a unique bounded centre-valued trace τ on A such that $D = d + \tau$.*

Proof. Applying Theorem 3.6 to the second adjoint D^{**} of D, we obtain the standard form of the Lie derivation D^{**} on the von Neumann algebra A^{**}, $D^{**} = \overline{d} + \overline{\tau}$ for a derivation $\overline{d}$ on A^{**} and a bounded centre-valued trace $\overline{\tau}$ on A^{**}. Since, for each $x \in A$, $(\overline{d} + \overline{\tau})(x) = D(x) \in A$, the preceding lemma shows that $\overline{d}(A) \subseteq A$ and $\overline{\tau}(A) \subseteq Z(A)$. Thus we can define $d = \overline{d}_{|A}$ and $\tau = \overline{\tau}_{|A}$ to obtain the standard form of D. $\square$

The remainder of this paper is devoted to verifying the criterion for the existence of a global centre-valued trace associated with a Lie derivation without additional assumptions, that is, to establish the continuity of the mapping $P \mapsto \tau_P(a)$ for all $a \in A$ (Proposition 3.3). To this end, it suffices to consider a unital C^*-algebra A and a Lie $*$-derivation D on A. Indeed, the usual decomposition of an arbitrary Lie derivation D into its real part $D_1 = \frac{1}{2}(D + D^*)$ and $D_2 = \frac{1}{2i}(D - D^*)$, where $D^*(x) = D(x^*)^*$ for each $x \in A$, implies that a standard form for D_1 and D_2 yields a standard form for D as well.

We shall consider $\mathrm{Prim}(A)$ endowed with its natural topology, the Jacobson or hull-kernel topology. Let

$$\mathfrak{G} = \{P \in \mathrm{Prim}(A) \mid A/P \ncong \mathbb{C}\}$$

and

$$\mathfrak{H} = \mathrm{Prim}(A) \backslash \mathfrak{G} = \{P \in \mathrm{Prim}(A) \mid A/P \cong \mathbb{C}\}.$$

Lemma 3.11. $\mathfrak{G}$ *is an open subset of* $\mathrm{Prim}(A)$.

Proof. Let $H = \bigcap_{P \in \mathfrak{H}} P$. It is easy to check that $\mathfrak{H} = \{P \in \mathrm{Prim}(A) \mid H \subseteq P\}$ and so $\mathfrak{H}$ is closed. $\square$

Lemma 3.12. *For every $a \in A$, the function $P \mapsto \tau_P(a)$ is continuous on $\mathfrak{G}$.*

Proof. Let $a \in A$, and let F be a closed subset of $\mathbb{C}$. Suppose that $Q \in \mathfrak{G}$ is such that

$$\bigcap_{\substack{P \in \mathfrak{G}, \\ \tau_P(a) \in F}} P \subseteq Q.$$

We only need to show that $\tau_Q(a) \in F$.

Since A/Q is non-commutative, there is $x \in A$ such that $x \notin Q$ and $x^2 \in Q$ (see, e.g., [2, Lemma 6.1.3] or [9, Exercise 4.6.30]). Letting $y = xx^*$ we find that

$$[[x, y], y] = [[x, xx^*], xx^*] \notin Q,$$

and hence $b=[[x,y],y]\,[[x,y],y]^* \notin Q$. Since $b=[[x,y],y]\,[y^*,[y^*,x^*]]$, Proposition 3.5 shows that $b\in A_D$. We also have

$$ab=[a[x,y],y]\,[y^*,[y^*,x^*]]-[a,y]\,[x,y]\,[y^*,[y^*,x^*]]$$

and Proposition 3.5 again yields that $ab\in A_D$.

Consider the abelian C^*-algebra

$$C(b)=\{u\in A\mid [u,v]=0 \text{ whenever } v\in A \text{ is such that } [b,v]=0\}$$

containing b and 1, and let

$$c=D(a)b+aD(b)-D(ab)-\tau(b)a+\tau(ab).$$

From Lemma 3.4(ii) we deduce that

$$\tau_P(a)\pi_P(b)=\pi_P(c)\quad (P\in \mathrm{Prim}(A))$$

and hence $c\in C(b)$. Indeed, if $v\in A$ is such that $[b,v]=0$, then

$$\pi_P([c,v])=[\pi_P(c),\pi_P(v)]=[\tau_P(a)\pi_P(b),\pi_P(v)]=\tau_P(a)\pi_P([b,v])=0$$

for each $P\in \mathrm{Prim}(A)$ and so $[c,v]=0$. Moreover, we have

$$\bigcap_{\substack{P\in\mathfrak{G},\\ \pi_P(c)\in F\pi_P(b)}} P \subseteq \bigcap_{\substack{P\in\mathfrak{G},\\ \tau_P(a)\in F}} P \subseteq Q.$$

Since $Q\cap C(b)=\bigcap\{N\in \mathrm{Prim}(C(b))\mid Q\cap C(b)\subseteq N\}$ and $b\notin Q$, there exists $N\in \mathrm{Prim}(C(b))$ such that $Q\cap C(b)\subseteq N$ and $b\notin N$. For every $P\in \mathrm{Prim}(A)$, we have

$$P\cap C(b)=\bigcap_{\substack{M\in \mathrm{Prim}(C(b)),\\ P\cap C(b)\subseteq M}} M.$$

Moreover, if $P\cap C(b)\subseteq M$ and $\pi_P(c)\in F\pi_P(b)$, then $\pi_M(c)\in F\pi_M(b)$, where π_M now stands for the quotient map from $C(b)$ onto $C(b)/M$. Consequently,

$$\bigcap_{\pi_M(c)\in F\pi_M(b)} M\subseteq \bigcap_{\pi_P(c)\in F\pi_P(b)} (P\cap C(b))\subseteq Q\cap C(b)\subseteq N,$$

where, for the rest of the proof, we will use the convention that M denotes primitive (i.e., maximal) ideals of $C(b)$ and P primitive ideals of A contained in $\mathfrak{G}$. Since $b\notin N$, we have

$$\bigcap_{\substack{\pi_M(c)\in F\pi_M(b),\\ b\notin M}} M\subseteq N.$$

Viewing b and c as continuous functions on $\mathrm{Prim}(C(b))$, the function c/b is continuous on $G = \{M \mid \pi_M(b) \neq 0\}$, and the set

$$\{M \in G \mid \pi_M(c) \in F\pi_M(b)\} = \{\omega \in G \mid c(\omega)/b(\omega) \in F\}$$

is closed in G. Since $N \in G$, it follows that $\pi_N(c) \in F\pi_N(b)$ and there is $\lambda \in F$ such that $c - \lambda b \in N$.

On the other hand,

$$\tau_Q(a)b - c \in Q \cap C(b) \subseteq N$$

and therefore $\tau_Q(a)b - \lambda b \in N$. Since $b \notin N$, it follows that $\tau_Q(a) = \lambda \in F$, as required. $\square$

This lemma in conjunction with Proposition 3.3 allows us to conclude from Theorem 3.1 the following result.

Corollary 3.13. *Let A be a unital C^*-algebra without characters. Then every Lie derivation D on A is of the form $D = d + \tau$ for a unique derivation d on A and a unique centre-valued trace τ.*

Lemma 3.14. *For every $a \in A$, the function $P \mapsto \tau_P(a)$ is continuous when restricted to $\mathfrak{H}$.*

Proof. First suppose that $a \in A$ is self-adjoint. Making use of our additional assumption that D is self-adjoint, the bicommutant $C(D(a))$ of $D(a)$ in A is an abelian C^*-algebra containing $D(a)$ and 1. We will use the convention that P stands for a primitive ideal in $\mathfrak{H}$ and M for a primitive ideal of $C(D(a))$.

Let F be a closed subset of $\mathbb{C}$ and suppose that $Q \in \mathfrak{H}$ is such that

$$\bigcap_{\tau_P(a) \in F} P \subseteq Q.$$

We only need to show that $\tau_Q(a) \in F$.

For every $P \in \mathfrak{H}$ we have $A/P \cong \mathbb{C}$ and therefore $d_P = 0$. Consequently, $\tau_P(a) = \pi_P(D(a))$ for each $P \in \mathfrak{H}$ and thus

$$\bigcap_{\pi_P(D(a)) \in F} P = \bigcap_{\tau_P(a) \in F} P \subseteq Q$$

and

$$\bigcap_{\pi_M(D(a)) \in F} M \subseteq \bigcap_{\pi_P(D(a)) \in F} (P \cap C(D(a))) \subseteq Q \cap C(D(a)).$$

Let $N \in \mathrm{Prim}(C(D(a)))$ be such that $Q \cap C(D(a)) \subseteq N$. Then

$$\bigcap_{\pi_M(D(a)) \in F} M \subseteq N.$$

Viewing $D(a)$ and 1 as continuous functions on $\mathrm{Prim}(C(D(a)))$, the set

$$\{M \in \mathrm{Prim}(C(D(a))) \mid \pi_M(D(a)) \in F\} = \{\omega \in \mathrm{Prim}(C(D(a))) \mid D(a)(\omega) \in F\}$$

is closed in $\mathrm{Prim}(C(D(a)))$, and therefore $\pi_N(D(a)) \in F$. Thus there is $\lambda \in F$ such that $D(a) - \lambda \in N$.

On the other hand,

$$\tau_Q(a) - D(a) \in Q \cap C(b) \subseteq N$$

and therefore $\tau_Q(a) - \lambda \in N$. Hence $\tau_Q(a) = \lambda \in F$, as required.

Now suppose that a is an arbitrary element in A. Then there exist $h, k \in A$ with $h^* = h$ and $k^* = k$ such that $a = h + ik$. We have $\tau_P(a) = \tau_P(h) + i\tau_P(k)$ for each $P \in \mathrm{Prim}(A)$. The first part of the proof shows that the maps $P \mapsto \tau_P(h)$ and $P \mapsto \tau_P(k)$ when restricted to $\mathfrak{H}$ are continuous and therefore the map $P \mapsto \tau_P(a)$ when restricted to $\mathfrak{H}$ is continuous. $\square$

If $\mathfrak{H}$ is open, which is the case when the ideals of codimension one are isolated points in $\mathrm{Prim}(A)$, then the map $P \mapsto \tau_P(a)$ is continuous for each $a \in A$. In the general situation, we have to do some more work. To this end, let $\mathfrak{C}$ be the set of those $Q \in \mathrm{Prim}(A)$ such that the map $P \mapsto \tau_P(a)$ is continuous at Q for all $a \in A$, and let

$$I = \bigcap_{P \in \mathfrak{C}} P \quad \text{and} \quad J = \bigcap_{P \in \mathfrak{D}} P,$$

where $\mathfrak{D} = \mathrm{Prim}(A) \setminus \mathfrak{C}$ and we write $J = A$ in the case where $\mathfrak{D} = \emptyset$.

Lemma 3.15. *The ideal J is essential and $I = 0$.*

Proof. Suppose that K is a closed ideal of A such that $J \cap K = 0$. The set $H = \{P \in \mathrm{Prim}(A) \mid K \nsubseteq P\}$ is an open subset of $\mathrm{Prim}(A)$. Moreover, if $P \in H$, then $J \subseteq P$. Indeed, $J \cap K = 0 \subseteq P$ and P is a prime ideal. Consequently, $H \subseteq \mathfrak{H}$ and Lemma 3.14 now shows that $H \subseteq \mathfrak{C}$. This clearly entails that every $P \in \mathfrak{D}$ satisfies $K \subseteq P$. Consequently,

$$K = \bigcap_{K \subseteq P} P \subseteq \bigcap_{P \in \mathfrak{D}} P = J.$$

Hence $0 = J \cap K = K$. This proves that J is essential.

Suppose that the set $G = \{P \in \mathrm{Prim}(A) \mid I \nsubseteq P\}$ is non-empty. Let $Q \in G$; as $Q \notin \mathfrak{C}$, there exists $a \in A$ such that the map $P \mapsto \tau_P(a)$ is discontinuous at Q. By Lemma 3.12

it follows that $Q \in \mathfrak{H}$. Thus, $G \subseteq \mathfrak{H}$ and Lemma 3.14 implies that the map $P \mapsto \tau_P(a)$ is continuous for each $a \in A$ when restricted to G. Since G is open, we conclude that $G \subseteq \mathfrak{C}$, a contradiction. As a result, G must be empty and thus $I = 0$. $\square$

The next result is the key to solving the remaining problem, since it will enable us to obtain the standard form of the Lie derivation on the essential closed ideal J.

Lemma 3.16. *Suppose that the function $P \mapsto \tau_P(a)$ is bounded for each $a \in A$. Then the function $P \mapsto \tau_P(a)$ is continuous on* $\mathrm{Prim}(A)$ *for all $a \in J$.*

Proof. It suffices to prove the assertion for each $a \in J$ which is positive. Let F be a closed subset of $\mathbb{C}$, and suppose that $Q \in \mathrm{Prim}(A)$ is such that

$$\bigcap_{\tau_P(a) \in F} P \subseteq Q.$$

We first consider the case in which $Q \in \mathfrak{C}$. In this case

$$\bigcap_{\substack{P \in \mathfrak{G}, \\ \tau_P(a) \in F}} P \subseteq Q,$$

which shows that Q is contained in the closure in $\mathrm{Prim}(A)$ of the set $\{P \mid \tau_P(a) \in F\}$. Since the map $P \mapsto \tau_P(a)$ is continuous at Q, it follows that $\tau_Q(a)$ belongs to the closure in $\mathbb{C}$ of the set $\{\tau_P(a) \mid P \in \mathrm{Prim}(A),\ \tau_P(a) \in F\}$, which is certainly contained in F.

We now suppose that $Q \in \mathfrak{D}$ and thus $J \subseteq Q$. Consider the bicommutant $C(a)$ of a in A and let

$$c = D(a) - D(a^{1/2})a^{1/2} - a^{1/2}D(a^{1/2}).$$

From Lemma 3.4(ii) we deduce that

$$\tau_P(a) - 2\tau_P(a^{1/2})\pi_P(a^{1/2}) = \pi_P(c)$$

for each $P \in \mathrm{Prim}(A)$, whence we obtain that $c \in C(a)$. Since the map $P \mapsto \tau_P(a^{1/2})$ is bounded, we can choose a compact subset K of $\mathbb{C}$ with the property that $-2\tau_P(a^{1/2}) \in K$ for each $P \in \mathrm{Prim}(A)$. We have

$$\bigcap_{\pi_P(c) \in F + K\pi_P(a^{1/2})} P \subseteq \bigcap_{\tau_P(a) \in F} P \subseteq Q$$

and therefore

$$\bigcap_{\pi_P(c) \in F + K\pi_P(a^{1/2})} (P \cap C(a)) \subseteq Q \cap C(a).$$

This shows that the intersection of all primitive (= maximal) ideals M of $C(a)$ such that $\pi_M(c) \in F + K\pi_M(a^{1/2})$ is contained in every primitive ideal N of $C(a)$ containing $Q \cap C(a)$.

Since the set $\{M \in \mathrm{Prim}(C(a)) \mid \pi_M(c) \in F + K\pi_M(a^{1/2})\}$ is closed in $\mathrm{Prim}(C(a))$, it follows that $\pi_N(c) \in F + K\pi_N(a^{1/2})$. Thus there exist $\lambda \in F$ and $\mu \in K$ with the property that $c - \lambda - \mu a^{1/2} \in N$. This shows that $c - \lambda \in N$, since $a^{1/2} \in J \subseteq Q$. On the other hand, $\tau_Q(a) - 2\,\tau_Q(a^{1/2})a^{1/2} - c \in Q \cap C(a) \subseteq N$ showing that $\tau_Q(a) - c \in N$. We thus conclude that $\tau_Q(a) = \lambda \in F$, as required. $\square$

Corollary 3.17. *Suppose that the function $P \mapsto \tau_P(a)$ is bounded for each $a \in A$. Then there exist a derivation $d : J \to J$ and a linear mapping $\tau : J \to Z(A)$ such that $D(a) = d(a) + \tau(a)$ for each $a \in J$.*

Proof. Combining Lemma 3.16 with Propositions 3.3 and 3.5 we find that $J \subseteq A_D$. The restriction of the derivation d defined on A_D to the closed ideal J leaves J invariant, which yields the assertion. $\square$

We now link the standard form of the Lie derivation on A with the ideal J and make a further reduction on the continuity problem.

Proposition 3.18. *The following assertions are equivalent.*

(a) *There exist a derivation d and a centre-valued trace τ on A such that $D = d + \tau$.*
(b) *For every $a \in A$, the function $P \mapsto \tau_P(a)$ is continuous on* $\mathrm{Prim}(A)$.
(c) *For every $a \in A$, the function $P \mapsto \tau_P(a)$ is bounded on* $\mathrm{Prim}(A)$.
(d) *For every $a \in A$, the function $M \mapsto \tau_M(a)$ is bounded on* $\mathrm{Max}(A)$, *the set of all maximal ideals of A.*

Proof. The equivalence of (a) and (b) follows from Propositions 3.3 and 3.5, while implication (b) $\Rightarrow$ (c) is a consequence of the compactness of $\mathrm{Prim}(A)$.

In order to establish implication (c) $\Rightarrow$ (a), let d and τ be given by Corollary 3.17. Let d also denote the extended derivation on $M(J)$, and note that $A \subseteq M(J)$, since J is essential. We claim that $D - d$ maps into the centre of $M(J)$. Indeed, for all $a \in A$ and $x \in J$, we have

$$[(D-d)(a), x] = (D-d)([a,x]) - [a, (D-d)(x)]$$
$$= (D-d)([a,x]) - [a, \tau(x)] = (D-d)([a,x]).$$

Since $[a,x] \in J$, we have $(D-d)([a,x]) = \tau([a,x])$. On the other hand, we have $\pi_P(\tau([a,x])) = \tau_P([a,x]) = 0$ for each $P \in \mathrm{Prim}(A)$ and therefore $\tau([a,x]) = 0$. This shows that $[(D-d)(a), x] = 0$. Thus $(D-d)(a)$ commutes with every element in J and therefore with every element in $M(J)$. By means of this, we can extend $\tau : J \to Z(A)$ to a mapping $D - d : A \to Z(M(J))$, which we also denote by τ.

We now claim that d maps A into itself and therefore $\tau(A) \subseteq Z(A)$. Indeed, $M(J)$ is a Banach A-bimodule and A is a closed subbimodule of $M(J)$. Since $d + \tau = D$ maps A into A and $\tau(A) \subseteq Z(M(J))$, Lemma 3.9 entails that $d(A) \subseteq A$.

Implication (c) $\Rightarrow$ (d) is trivial, and the converse implication is a consequence of Lemma 3.4(i) together with the fact that every primitive ideal of A is contained in a maximal ideal since A is unital. $\square$

The final lemma establishes condition (d) in the above proposition.

Lemma 3.19. *For every $a \in A$, the function $M \mapsto \tau_M(a)$ is bounded on* $\mathrm{Max}(A)$.

Proof. Since A/M is a unital simple C^*-algebra, the derivation d_M given by Theorem 3.1 is inner (see, e.g., [2, Corollary 4.2.9] or [9, Exercise 10.5.78]). Suppose first there is a tracial state f_M of A/M. Since d_M is inner, it follows that $f_M \circ d_M = 0$. Consequently, for every $a \in A$, we have

$$\tau_M(a) = f_M(\tau_M(a)\pi_M(1)) = f_M(\pi_M(D(a))) - f_M(d_M(\pi_M(a))) = f_M(\pi_M(D(a))),$$

which shows that $|\tau_M(a)| \leqslant ||D(a)||$.

Assume now that the lemma fails to be true, and let $a \in A$ be such that the function $M \mapsto \tau_M(a)$ is unbounded. By the first paragraph of this proof, we can inductively define a sequence $(M_n)_{n \in \mathbb{N}}$ in $\mathrm{Max}(A)$ such that codim M_n is infinite and

$$|\tau_{M_{n+1}}(a)| > 1 + |\tau_{M_n}(a)|$$

for each $n \in \mathbb{N}$. In particular, we have

$$||\tau_{M_i}(a)| - |\tau_{M_j}(a)|| > 1 \quad \text{if } i \neq j.$$

Note that

$$\bigcap_{k \neq n} M_k \nsubseteq M_n.$$

Indeed, if $\bigcap_{k \neq n} M_k \subseteq M_n$, then M_n is in the closure in $\mathfrak{G}$ of the set $\{M_k \mid k \neq n\}$. Since the function $P \mapsto \tau_P(a)$ is continuous when restricted to $\mathfrak{G}$ (Lemma 3.12), it follows that $\tau_{M_n}(a)$ is in the closure in $\mathbb{C}$ of the set $\{\tau_{M_k}(a) \mid k \neq n\}$. But this is impossible, since

$$|\tau_{M_k}(a) - \tau_{M_n}(a)| \geqslant ||\tau_{M_k}(a)| - |\tau_{M_n}(a)|| > 1$$

if $k \neq n$.

Let $n \in \mathbb{N}$. Since $\bigcap_{k \neq n} M_k \nsubseteq M_n$, there exists $u_n \in \bigcap_{k \neq n} M_k$ such that $\pi_{M_n}(u_n) = \pi_{M_n}(1)$ and $||u_n|| = ||\pi_{M_n}(u_n)||$. Setting $I = \bigcap_{n=1}^{\infty} M_n$ we thus find that

$$\bigoplus_{n=1}^{\infty} A/M_n \subseteq A/I \subseteq \prod_{n=1}^{\infty} A/M_n,$$

where we canonically identify the coset $x+I$ in A/I with the sequence $(x+M_n)_{n\in\mathbb{N}}$ in $\prod_{n=1}^{\infty} A/M_n$ and thus can consider each algebra A/M_n inside of A/I via $1+M_n \mapsto (u_n+M_k)_{k\in\mathbb{N}}$.

Put $B=A/I$ and note that, since $\dim A/M_n > 4$ for all n, the simplicity of A/M_n implies that $K_2(A/M_n)=A/M_n$. Thus, $K_2^{\perp}(B)=0$ and we can apply Theorem 2.1. From this, we obtain a derivation $d_I : A \to {}^{c}B$ and a linear mapping $\tau_I : A \to Z({}^{c}B)$ such that $D_I = d_I + \tau_I$. (We note in passing the extra piece of information that ${}^{c}B = \prod_{n=1}^{\infty} A/M_n$, by Ara and Mathieu [2, Propositions 2.3.6 and 3.3.7].) Lemma 3.9 entails that, in fact, $d_I(A) \subseteq B$ and $\tau_I(A) \subseteq Z(B)$.

For each $n\in\mathbb{N}$, let π_n be the canonical mapping from A/I onto A/M_n. The identity $D_I = d_I + \tau_I$ yields $D_{M_n} = \pi_n d_I + \pi_n \tau_I$. Note that $\pi_n d_I$ is a derivation from A into A/M_n and that $\pi_n \tau_I$ maps into $Z(A/M_n)$. Corollary 2.2 shows that $d_{M_n} = \pi_n d_I$ and thus $\|d_{M_n}\| = \|\pi_n d_I\| \leqslant \|d_I\|$ for all $n\in\mathbb{N}$. Hence

$$|\tau_{M_n}(x)| \leqslant \|D(x)\| + \|d_{M_n}(x)\| \leqslant \|D(x)\| + \|d_I\|\,\|x\| \quad (n\in\mathbb{N},\ x\in A),$$

contradicting the unboundedness of $(\tau_{M_n}(a))_{n\in\mathbb{N}}$. □

Proof of Theorem 1.1. Let D be a Lie derivation on the C^*-algebra A. If A is non-unital, we extend D to a Lie derivation D_1 on the unitisation A_1 of A by defining $D_1(1)=0$. Suppose that $D_1 = d_1 + \tau_1$, where d_1 is a derivation on A_1 and τ_1 is a linear mapping from A_1 into its centre. Since A is a closed ideal in A_1, it follows that $d_1(A) \subseteq A$ and so $\tau_1(A) \subseteq Z(A)$. Letting d and τ be the restriction of d_1 and τ_1 to A, respectively, we thus obtain the standard decomposition $D = d + \tau$. Therefore, we may assume that A is unital.

We can now apply Lemma 3.19 and Proposition 3.18 to obtain the standard form of D, and the proof is complete. □

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